SECTION 5.5: SUBSTITUTION

RECALL: The chain rule: $D_x[F(g(x))] = F'(g(x))\, g'(x)$.

Hence, if $F'(x) = f(x)$, then $\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$.

If above we let $u = g(x)$, then $du = g'(x)\, dx$ and we have:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

This is the theory behind 'u-substitution.' Substitution can be thought of as reversing the chain rule.

EXAMPLE 1: Find $\int (x^2+5)^{117}\, 2x\, dx$.

We let $u = x^2 + 5$ so $du = 2x\, dx$.

Then the integral becomes: $\int (x^2+5)^{117}\, 2x\, dx = \int u^{117}\, du = \frac{1}{118} u^{118} + C = \frac{1}{118}(x^2+5)^{118} + C$

As always, we check our answer by taking the derivative. Notice the chain rule applies!

$$D_x\left[\frac{1}{118}(x^2+5)^{118} + C\right] = \frac{1}{118}\left(118\left(x^2+5\right)^{118-1} D_x\left[x^2+5\right]\right) + 0 = \left(x^2+5\right)^{117} 2x \checkmark$$

EXAMPLE 2: Find $\int \frac{\sec^2(\theta)}{\sqrt{1+\tan(\theta)}}\, d\theta$ using the substitution $u = 1 + \tan(\theta)$.

Check your answer using differentiation.

If $u = 1 + \tan(\theta)$ then $du = \sec^2(\theta)\, d\theta$. Hence:

$$\int \frac{\sec^2(\theta)}{\sqrt{1+\tan(\theta)}}\, d\theta = \int (1+\tan(\theta))^{-\frac{1}{2}} \sec^2(\theta)\, d\theta = \int u^{-\frac{1}{2}}\, du = 2\, u^{\frac{1}{2}} + C = 2\,(1+\tan(\theta))^{\frac{1}{2}} + C$$

To check, we compute: $D_\theta\left[2\,(1+\tan(\theta))^{\frac{1}{2}} + C\right] = 2\left(\frac{1}{2}\right)(1+\tan(\theta))^{-\frac{1}{2}}\, D_\theta[1+\tan(\theta)] + 0 = \frac{\sec^2(\theta)}{\sqrt{1+\tan(\theta)}} \checkmark$

NOTE: Since substitution reverses the chain rule: $D_x[f(u)] = f'(u)\, u'$, we often try letting u be whatever function appears to be 'inside' another function. Less formally:

RULE OF THUMB: Let $u =$ whatever is in parentheses.

EXAMPLE 3: Find $\displaystyle\int \frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}\,dt$.

We first try $u = \sqrt{t}$ since $\sqrt{t}$ is the 'inside' function of the composition: $\sin\left(\sqrt{t}\right)$. We find $du = \dfrac{1}{2}t^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{t}}$.

Rewriting the integral as: $\displaystyle\int \frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}\,dt = \int \sin\left(\sqrt{t}\right)\frac{1}{\sqrt{t}}\,dt$, we see the factor $\dfrac{1}{\sqrt{t}}\,dt$ almost matches our du.

To get it to match exactly, we multiply the integrand by $1 = \dfrac{2}{2}$ so as to not change the value of the integrand:

$$\begin{aligned}
\int \sin\left(\sqrt{t}\right)\frac{1}{\sqrt{t}}\,dt &= \int \sin\left(\sqrt{t}\right)\frac{2}{2\sqrt{t}}\,dt \\
&= 2\int \sin\left(\sqrt{t}\right)\frac{1}{2\sqrt{t}}\,dt \qquad \text{Constant Multiple Rule} \\
&= 2\int \sin(u)\,du = -2\cos(u) + C \\
&= -2\cos\left(\sqrt{t}\right) + C
\end{aligned}$$

As always, we check: $D_t\left[-2\cos\left(\sqrt{t}\right) + C\right] = -2\left(-\sin\left(\sqrt{t}\right)D_t\left[\sqrt{t}\right]\right) + 0 = \cdots = \dfrac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}$ ✓

EXAMPLE 4: Find $\displaystyle\int \theta\sec\left(\theta^2\right)\tan\left(\theta^2\right)d\theta$. Check your answer using differentiation.

We let $u = \theta^2$ so $du = 2\theta\,d\theta$. Rewriting the integrand to account for the factor of '2' in the du, we get:

$$\begin{aligned}
\int \theta\sec\left(\theta^2\right)\tan\left(\theta^2\right)d\theta &= \int \sec\left(\theta^2\right)\tan\left(\theta^2\right)\theta\,d\theta \\
&= \int \sec\left(\theta^2\right)\tan\left(\theta^2\right)\frac{2}{2}\theta\,d\theta \\
&= \frac{1}{2}\int \sec\left(\theta^2\right)\tan\left(\theta^2\right)2\theta\,d\theta \\
&= \frac{1}{2}\int \sec(u)\tan(u)\,du \\
&= \frac{1}{2}\sec(u) + C \\
\int \theta\sec\left(\theta^2\right)\tan\left(\theta^2\right)d\theta &= \frac{1}{2}\sec\left(\theta^2\right) + C
\end{aligned}$$

To check, we find:

$$D_\theta\left[\frac{1}{2}\sec\left(\theta^2\right) + C\right] = \frac{1}{2}\sec\left(\theta^2\right)\tan\left(\theta^2\right)D_\theta\left[\theta^2\right] + 0 = \frac{1}{2}\sec\left(\theta^2\right)\tan\left(\theta^2\right)(2\theta) = \theta\sec\left(\theta^2\right)\tan\left(\theta^2\right) ✓$$

EXAMPLE 5: Find $\displaystyle\int_5^{13} \sqrt{2x-1}\,dx$.

Rewriting $\displaystyle\int_5^{13} \sqrt{2x-1}\,dx = \int_5^{13} (2x-1)^{\frac{1}{2}}\,dx$, we let $u = 2x-1$ so that $du = 2\,dx$.

To match the du exactly, we rewrite the integral as: $\displaystyle\int_5^{13} (2x-1)^{\frac{1}{2}}\,\frac{2}{2}\,dx = \frac{1}{2}\int_5^{13} (2x-1)^{\frac{1}{2}}\,2\,dx$

Since we have a definite integral, we **<u>need to convert the limits of the integral</u>** from x-values to u-values.

Since $u = 2x-1$, when $x = 5$, $u = 2(5)-1 = 9$. Likewise, when $x = 13$, $u = 2(13)-1 = 25$. Hence:

$$\frac{1}{2}\int_5^{13} (2x-1)^{\frac{1}{2}}\,2\,dx = \frac{1}{2}\int_9^{25} u^{\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{u=9}^{u=25} = \frac{1}{3}(25)^{\frac{3}{2}} - \frac{1}{3}(9)^{\frac{3}{2}} = \ldots = \frac{98}{3}$$

EXAMPLE 6: Find $\displaystyle\int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \theta^{-2}\cos\left(\frac{1}{\theta}\right)\,d\theta$.

We let $u = \dfrac{1}{\theta} = \theta^{-1}$ so that $du = (-1)\theta^{-2}\,d\theta$. When $\theta = \frac{2}{\pi}$, $u = \frac{\pi}{2}$ and when $\theta = \frac{1}{\pi}$, $u = \pi$. Hence:

$$\begin{aligned}
\int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \theta^{-2}\cos\left(\frac{1}{\theta}\right)\,d\theta &= \int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \cos\left(\frac{1}{\theta}\right)\,\theta^{-2}\,d\theta \\
&= \int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \cos\left(\frac{1}{\theta}\right)\frac{(-1)}{(-1)}\theta^{-2}\,d\theta \\
&= -\int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \cos\left(\frac{1}{\theta}\right)(-1)\theta^{-2}\,d\theta \\
&= -\int_{\frac{\pi}{2}}^{\pi} \cos(u)\,du \\
&= -\sin(u)\big|_{u=\frac{\pi}{2}}^{u=\pi} \\
&= \left(-\sin\left(\pi\right)\right) - \left(-\sin\left(\tfrac{\pi}{2}\right)\right) \\
&= 0 + 1
\end{aligned}$$

Hence, $\displaystyle\int_{\frac{2}{\pi}}^{\frac{1}{\pi}} \theta^{-2}\cos\left(\frac{1}{\theta}\right)\,d\theta = 1$.

EXAMPLE 7: Find $\int_1^{10} x\sqrt{x-1}\,dx$.

Writing $\int_1^{10} x\sqrt{x-1}\,dx = \int_1^{10} x(x-1)^{\frac{1}{2}}\,dx$, we let $u = x-1$ so $du = dx$.

Converting the limits, we get when $x = 1$, $u = 1-1 = 0$ and when $x = 10$, $u = 10-1 = 9$.

When we go to make our substitution, however, we have an x remaining: $\int_1^{10} x(x-1)^{\frac{1}{2}}\,dx = \int_0^9 x\,u^{\frac{1}{2}}\,du$.

To fully convert this integral in terms of u, we solve the substitution equation $u = x-1$ for x to get $x = u+1$.

Hence, we evaluate the integral as follows:

$$\begin{aligned}
\int_1^{10} x(x-1)^{\frac{1}{2}}\,dx &= \int_0^9 (u+1)\,u^{\frac{1}{2}}\,du \\
&= \int_0^9 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)\,du && \text{distribute} \\
&= \int_0^9 u^{\frac{3}{2}}\,du + \int_0^9 u^{\frac{1}{2}}\,du && \text{sum rule} \\
&= \frac{2}{5}u^{\frac{5}{2}}\Big|_{u=0}^{u=9} + \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=0}^{u=9} \\
&= \frac{486}{5} - 0 + 18 - 0 = \ldots = \frac{576}{5}
\end{aligned}$$

EXAMPLE 8: Find $\displaystyle\int \frac{x}{(2x+1)^3}\,dx$. Check your answer using differentiation.

We let $u = (2x+1)$ so that $du = 2\,dx$ and $x = \dfrac{u-1}{2}$. We get:

$$\begin{aligned}
\int \frac{x}{(2x+1)^3}\,dx &= \int \frac{x}{(2x+1)^3}\,\frac{2}{2}\,dx \\
&= \frac{1}{2}\int \frac{x}{(2x+1)^3}\,2\,dx \\
&= \frac{1}{2}\int \frac{\left(\dfrac{u-1}{2}\right)}{u^3}\,du \\
&= \frac{1}{2}\int \left(\frac{u-1}{2}\right)\,u^{-3}\,du \\
&= \frac{1}{4}\int (u-1)\,u^{-3}\,du \\
&= \frac{1}{4}\int \left(u^{-2}-u^{-3}\right)\,du \\
&= \frac{1}{4}\int u^{-2}\,du - \frac{1}{4}\int u^{-3}\,du \\
&= -\frac{1}{4}u^{-1} + \frac{1}{8}u^{-2} + C \\
\int \frac{x}{(2x+1)^2}\,dx &= -\frac{1}{4}(2x+1)^{-1} + \frac{1}{8}(2x+1)^{-2} + C
\end{aligned}$$

To check we find:

$$\begin{aligned}
D_x\left[-\frac{1}{4}(2x+1)^{-1} + \frac{1}{8}(2x+1)^{-2} + C\right] &= -\frac{1}{4}(-1)(2x+1)^{-2}D_x[2x+1] + \frac{1}{8}(-2)(2x+1)^{-3}D_x[2x+1] + 0 \\
&= \frac{1}{4}\left(\frac{1}{(2x+1)^2}\right)(2) - \frac{1}{4}\left(\frac{1}{(2x+1)^3}\right)(2) \\
&= \frac{1}{2\,(2x+1)^2} - \frac{1}{2\,(2x+1)^3} \\
&= \frac{(2x+1)-1}{2\,(2x+1)^3} \\
&= \frac{2x}{2\,(2x+1)^3} = \frac{x}{(2x+1)^3}\ \checkmark
\end{aligned}$$

HOMEWORK: Section 5.5: 1 - 65 odd, 75 - 81 odd, 91, 95*, 99*, 105, 113*

SUBSTITUTION EXTRA PRACTICE (VIDEO)

Find the following integrals.

1. $\displaystyle\int \sqrt{1-x}\,dx$

2. $\displaystyle\int x\sqrt{1-x^2}\,dx$

3. $\displaystyle\int x\sqrt{1-x}\,dx$

4. $\displaystyle\int \sin(x)\sqrt{1-\cos(x)}\,dx$

5. $\displaystyle\int \frac{3}{(x-1)^3}\,dx$

6. $\displaystyle\int \frac{3x}{(x-1)^5}\,dx$

7. $\displaystyle\int \frac{3x}{(x^2-1)^2}\,dx$

8. $\displaystyle\int \frac{3}{x^2-2x+1}\,dx$

   HINT: Factor...

9. $\displaystyle\int \frac{\sec(x)}{\tan^2(x)+1}\,dx$

   HINT: Identities...

10. $\displaystyle\int_0^{2\pi} \sqrt{1-\cos(x)}\,dx$

    HINT: $1-\cos(x) = 2\sin^2(x/2)\ \ldots$

ANSWERS:

1. $\displaystyle\int \sqrt{1-x}\, dx = -\frac{2}{3}\,(1-x)^{3/2} + C$

2. $\displaystyle\int x\sqrt{1-x^2}\, dx = -\frac{1}{3}\,(1-x^2)^{3/2} + C$

3. $\displaystyle\int x\sqrt{1-x}\, dx = \frac{2}{5}\,(1-x)^{5/2} - \frac{2}{3}\,(1-x)^{3/2} + C$

4. $\displaystyle\int \sin(x)\sqrt{1-\cos(x)}\, dx = \frac{2}{3}\,(1-\cos(x))^{3/2} + C$

5. $\displaystyle\int \frac{3}{(x-1)^3}\, dx = -\frac{3}{2}\,(x-1)^{-2} + C$

6. $\displaystyle\int \frac{3x}{(x-1)^5}\, dx = -(x-1)^{-3} - \frac{3}{4}\,(x-1)^{-4} + C$

7. $\displaystyle\int \frac{3x}{(x^2-1)^2}\, dx = -\frac{3}{2}\,(x^2-1)^{-1} + C$

8. $\displaystyle\int \frac{3}{x^2-2x+1}\, dx = \int \frac{3}{(x-1)^2}\, dx = -3\,(x-1)^{-1} + C$

9. $\displaystyle\int \frac{\sec(x)}{\tan^2(x)+1}\, dx = \int \frac{\sec(x)}{\sec^2(x)}\, dx = \int \cos(x)\, dx = \sin(x) + C$

10. $\displaystyle\int_0^{2\pi} \sqrt{1-\cos(x)}\, dx = \int_0^{2\pi} \sqrt{2\sin^2(x/2)}\, dx = \ldots = 4\sqrt{2}$